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The Optimal Trajectory in Top-Level Alpine Skiing[1]

Abstract

In order to investigate the relevance of Bernoulli's so-called „brachistochrone" theory experiments have been made with three international-level alpine ski-racers. The pure brachistochrone paradoxical effect, saying that the racing-time along a curved trajectory segment is shorter than along a corresponding straight segment, has been experimentally proved for racers starting out of a resting position. Since such an effect is typical of energy-conservative dynamical systems, we conclude that ski-racers with a top-level ski technique do not experience significant damping (i.e. do not significantly brake) during most of their transverse trajectories at low velocities (typically less than 40 km/h). The delicate point concerning air resistance at such velocities is discussed. We point out the following implications of our findings on the future strategy which should be adopted by top ski-racers in Giant or „Super-Giant: Super-G"-slaloms: extending Bernoulli's theory to non-zero initial velocities as well as taking into account the significant effect of the air resistance at such velocities (typically higher than 40 km/h), we predict that the trajectories in international Giant (or Super-G) slalom ski-races are expected to adapt themselves to what we call the „Going-Straight-Turning-Short" strategy. The biomechanical and technical problems related to such a resulting „Z" trajectory are pointed out.

Zusammenfassung

Zum Beweis der Bedeutung Bernoullis „Brachistochroner Theorie" für den Schirennsport wurden Feldexperimente mit drei Läufern internationalen Niveaus durchgeführt. Der reine „Brachistochrone Paradox-Effekt", der besagt, daß die schnellste Verbindung zweier Punkte auf einer ebenen geneigten Fläche entlang einer Kurve und nicht entlang der Geraden, die diese beiden Punkte verbindet, liegt, wurde experimentell unter der Bedingung, daß die Rennläufer aus der Ruhe starteten, nachgewiesen.

Aus diesem Effekt im Sinne eines energieerhaltenden Systems schließen wir, daß Spitzenläufer bei einer Geschwindigkeit unter 40 km/h während der meisten Schrägfahrten nicht langsamer werden. Die Frage nach der Bedeutung des Luftwiderstandes unter den Bedingungen niedriger Geschwindigkeiten wird diskutiert.

Auf Grund der Forschungsergebnisse werden folgende Vorschläge an den Hochleistungssport im Riesentorlauf und Super G herangetragen:

Überträgt man Bernoullis Theorie auf die Bewältigung der Rennstrecke nach dem Start und berücksichtigt dabei den signifikanten Effekt des Luftwiderstandes bei höheren Geschwindigkeiten als 40 km/h ergibt sich für den Läufer die „Z -Linie" des „Gerade Ansteuerns des Tores - schnelles Drehen um das Tor" als schnellste Verbindung zwischen zwei Toren. Die biomechanischen und technischen Probleme einer solchen Spurwahl werden ausgeführt.

1. Introduction

Recently, interest became to focus on the subject of alpine ski-racing trajectories since a three-century old scientific result was re-discovered in the context of ski competition

[1] Zu diesem Beitrag finden Sie ein Interpretorial auf Seite 95.

and translated into terms which are immediately available to ski coaches (Reinisch, 1988, 1990, 1991; Twardokens, 1988, 1990). The issue under consideration is the so-called *optimal* trajectory, which in a given race-course configuration *does* indeed exist *without* taking into account the present state of the art in alpine ski technique and/or technology. It exists because of the general laws of physics, and it exists in the sense of a mathematical limit: you know that you will never be able to reach it. It *should* however define the trend of your action.

Referring to the optimal trajectory theory, having been rediscovered in the context of skiracing in 1988, the question now arises whether this theory should define the trends of the future ski-technique and/or ski-training and ski-technology.

The theoretical problem which led to this theory presents itself in the following manner: A so-called „material point-particle" starting from zero movement at a given uphill point A, is designed to reach a given downhill point B within the shortest time possible, under the sole influence of a *constant* gravitational field g. The problem of finding the *optimal* trajectory under these terms has been solved nearly three centuries ago (1696) by J. Bernoulli and was given the rather strange-sounding Greek name: „brachistochrone" (which means: „the shortest time"). The solution is a portion of a „cycloid" (Goldstein, 1956) and is illustrated in the present case by Figure 1:

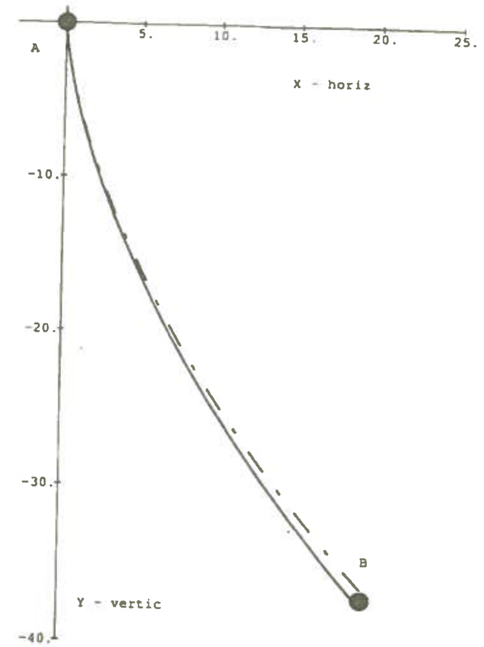

Figure 1

Figure 1: The map of the race-course: the racers start from standstill at the uphill point A and reach point B, whose horizontal and vertical co-ordinates (along the fall line) are: X = 19.7m; Y = 37.1m. An error of the order of 1m concerning the value of X has a weak effect on the optimal cycloid brachistochrone trajectory: indeed the dotted curve corresponds to X = 19.7m, Y = 37.1m, while the continuous curve corresponds to X = 18.7m and H = 37.1m. The extensions of the points A and B represent the localisation uncertainty (of order 1.2m) of these points. The angle of the slope equals 26 degrees.

It poses than no serious scientific difficulty to extend Bernoulli's theory - definitively called the „variational techniques" in mathematics - to more realistic cases including a non-zero initial velocity v_A at Point A. The result is still the portion of a cycloid whose „peak" now lies *above* the starting point A, at an altitude h_A so that the corresponding potential energy mgh_A equals the kinetic energy $(1/2)mv_A^2$ (Reinisch, 1990, 1991). Consequently, the corresponding trajectory AB appears more straight than for the zero-initial-velocity case. (see Figure 2)

Figure 2: The optimal trajectories AB corresponding to the skiers centre-of-gravity starting at Point A with decreasing initial velocities: v_A equals from bottom to top: 50 km/h, 40 km/h, 30 km/h, 20 km/h and 0 km/h (i.e. starting at rest: the brachistochrone problem). The slope's decline is 20 degrees for all cases. The values of x and y are given in meters and the scales of the two axes are identical. Each trajectory corresponds to the same segment defined by the couple of points A, B whose horizontal and vertical abcissa are : X = 7.5m and Y = 13m. The position of point A on the vertical y-axis corresponds to the potential energy of the skier at this point, namely: $y_A = v_A^2/2g$.

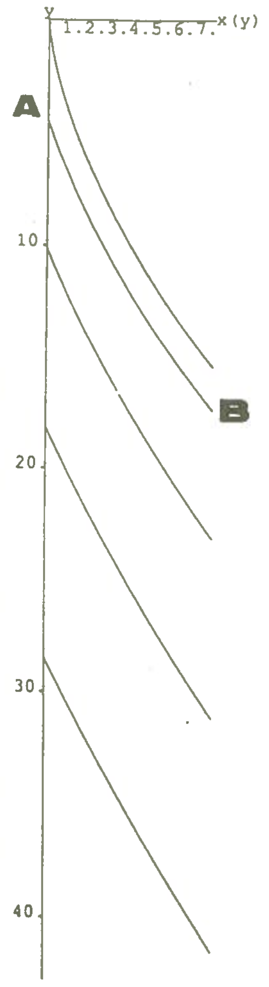

Figure 2

How may these results be adapted to alpine ski-racing, or more precisely, to Super-G (SG) or giant slalom (GS)?

1. First of all, we need a *uniform plane slope*. While this is not a serious problem for performing a given experiment - for the test of the „brachistochrone" theory, a 50m (horizontal length) by 100m (vertical length) snow plane is enough - the question arises whether such uniform plane slopes are available in realistic alpine ski-racing conditions (where non-zero initial-velocity values are of course required).

Remarkably enough, an international SG or GS race-course of World Cup level is often designed in such a way that it includes large parts (ten gates or even more) of rather uniform slopes.

2. Secondly, we need a *mass point*. According to the general laws of physics, it *is* the centre of mass of the skier. Then the trajectory of this point G_0 will remain reasonably close to the trajectory actually described by the centre of pressure P_0 of the steering-ski's curving edge, as long as the distance $d = G_0P_0$ between these two points (approximately 1 meter) is small compared to the characteristic length of the course. Obviously, this length is the spatial distance λ of the race-course, i.e. the average distance between two adjacent gates. Under SG or GS conditions, λ is bigger than or around 25 meters, while it is 10 meters or less for a slalom.

3. Thirdly - and this is the most important point - on one hand Bernoulli's theory is only relevant for *conservative dynamical systems* (as emphasized earlier), whereas with skiing two kinds of energy losses must obviously be taken into account:

- the ski-snow contact: The movement is most energy-conservative the better skiing technique of the racer is and the less severe turns the trajectory includes. This means that Bernoulli's theory *„a priori"* best applies for good ski racers performing transverse trajectory segments (of the above-defined length λ) which lie *between two gates* (i.e. between two turns, as severe energy losses obviously occur during the turns).
- the air resistance: It highly depends on the equipment. A Super Giant or Giant Slalom synthetic skiing suit, which is characteristic of the present-day top level skiing, is very

efficient in order to lower the air resistance. Nevertheless, air resistance may remain significant and cannot be simply ignored.

Let us emphasise a point which is important for the conclusions of the present work and should settle one's doubts about the validity of the brachistochrone theory referring to ski racing.

The air resistance depends on the square of the velocity. This velocity varies by a few percent when changing from a straight trajectory segment of length λ to the corresponding brachistochrone cycloid segment. For instance, the relative gain of racing-time which is in the present work is around 3%, leads to a relative increase of the air resistance amounting to 6%. When introducing such a variation of the damping factor in the kinetic and/or variational equations ascribed to the non-conservative brachistochrone problem (see Reinisch, 1990, 1991), one does not obtain significant changes in the definition of the optimal trajectory as long as *low velocities* (typically less than 40 km/h) are concerned. It still remains a (slightly flattened) cycloidlike trajectory segment instead of the straight line. These are the results of our present work, where the initial velocity of the racers is zero.

On the other hand, starting at non-zero velocities (typically higher than 40 km/h in order to describe relevant GS or SG conditions) leads to an extreme flatness of the optimal trajectory (Reinisch, 1990, 1991), *either by extending the original conservative Bernoulli theory in order to account such initial velocities or by taking into account the significant damping factor due to air resistance in the generalised non-conservative Bernoulli theory*. Actually both trajectories (conservative or not-conservative) are combined at high velocities when significant damping occurs (the shorter the trajectory, the smaller the energy loss by friction).

Therefore an account of the air resistance strengthens the conclusion that the optimal trajectory between two adjacent gates lies very close to the straight line as soon as the velocity exceeds *40 km/h*, i.e. in usual SG or GS racing conditions. Hence we obtain the so-called G.S.T.S. piloting strategy (going straight between two gates, turning short at the gates) which leads to a suggestion of Z-trajectories in SG or GS conditions (Reinisch, 1988, 1990, 1991).

Having stressed the theoretical interest which can be ascribed to the extended Bernoulli theory in the evolution of GS ski-racing technique, it is of crucial importance to check the practical relevance by experiments in snow. The non-plus-ultra-program would of course include the experimental check-up of all three above assumptions (1) to (3). This is scheduled for the near future, but we first found ourselves obliged to check, as a „must", the *pure brachistochrone effect for top-level ski-racers* (i.e. starting from a standstill), in order to verify that *they are indeed able to follow a transverse trajectory segment whose length is about λ without losing a significant amount of their total energy (kinetic + potential energy)*. As a result, they should obtain a shorter racing-time along a cycloid-like trajectory segment than along the *even shorter* corresponding straight line (see Figure 1).

Moreover, the impressive quantitative correlation of the *experimental results* with the theoretical predictions shows the adequacy of the very simple model used in the present study. Let us emphasise the drastic assumptions made in the present paper:

- air resistance and friction are both neglected because of the low velocities and the high technical level of the test skiers - i.e. the system is assumed conservative in the first approximation;
- the corresponding Bernoulli brachistochrone model is based on the movement of a single mass point. In the experiment it is represented by the skier's centre of gravity. Any biomechanical segmental and/or inertial effects which do obviously occur since a complex ski technique is involved are not taken into account.

The fact that such a simplified theoretical model works efficiently brings about the following important conclusion: *the ski technique of any high-level skier spontaneously adapts the motion of the skier´s centre-of-mass obeying simple physical laws.*
Therefore the study of this motion is the ground for future developments in skiing-technique.

The first attempt to check the validity of the brachistochrone theory in snow has been made by G. Twardokens in collaboration with the American Ski Association (Twardokens, 1988). In this experiment, the straight transverse segment and the optimal cycloid curve were marked in snow with a colour dye and the skiers were asked to ski along these trajectories. „*The experiment was successful in 14 out of 18 runs. Most of the time the cycloid curve produced shorter times than the straight line*", Twardokens wrote.
In this paper we give the account of a complementary experiment which:
a) is benefited from international-level ski-racers (actually between 40 and 60 FIS points). This was not the case in Twardoken´s experiment.
b) lets the racer choose his/her own „rather straight" or „downhill-curved" trajectory, comparing the corresponding racing-time with the appropriate theoretical result. Hence - and this is a very important point in our opinion - *the skiers were not forced to follow a previously marked trajectory*, which, we believe, would have altered their natural skiing-technique for the worse.
c) in order to provide the corresponding relevant theoretical results, the theory was extended to include the predicted racing-time values for a family of curves which range from the optimal brachistochrone cycloid to quasi-straight transverse trajectory segments.

2. The Experimental Set-up in Snow and the Corresponding Protocol (Tignes, France: Feb. 91)

Three international-level alpine ski-racers (FIS-point range: 40-60 points) belonging to the French University Ski-Team functioned as test-persons: two femal racers (Muriel Martinet: MM, and Martine Renaud-Goud: MRG) and one male racer (Jérémy Roch: JR).
A snow plane whose parameters are the following: vertical size: 60m; horizontal size: 40m; angle of slope θ such that $26^0 < \theta < 27^0$ was prepared on the Olympic acrobatic ski slope according to the usual requirements of international ski racing (hard snow; smooth surface). The weather was fine, the visibility excellent and there was no significant wind.
The racing-time values were recorded by the standard race-course stop-watch devises: electronic starting gate at the uphill point A, photocell at the downhill mark point B.

The parameters concerning the segment AB are the following: (see Figure 1)
horizonzal co-ordinate: $X = 19.7m \pm 0.5m$,
vertical co-ordinate (along the fall-line): $Y = 37.1m \pm 0.5m$ **(1)**

The racers were asked to start at point A from standstill, accelerated only by gravity, (hence without the help of their sticks) and to cross, between their skis, the mark point B located of course in the line-of-sight of the time-recording photo-cell and designed as a black rubber ball fixed on a small stick end, 0.10m above the snow surface.
The task of the first run was to go on an imaginary straight line as straight as possible from point A to point B. Then the racers were *qualitatively* told about the brachistochrone paradox, namely that a longer downhill-curved trajectory AB may result in a shorter time than the corresponding straight-transverse one. But, as they were now asked for the next run to „curve their trajectory downhill", they were *left free to choose the curvature for this new attempt.*
In order to avoid any learning effects from one performance to the next one and thus in order to obtain a sort of immediate qualitative, as well as quantitative, pictures of the racers natural reaction when facing the brachistochrone paradox, the racers were allowed only one attempt per task (i.e. an individually chosen straight trajectory and a curved one).
Taking into account the uncertainty in the initial (point A) and final (point B) direction of the racers´ trajectory with respect to the line-of-sight of the time-recording photo-cell, together with the small average velocity (4m/s or 15 km/h), with the width of the starting gate (on the order of 0.5m) and with the above-mentioned localisation process of the mark B, we obtain a potential error in each racing-timerecord of ± 0.02s.

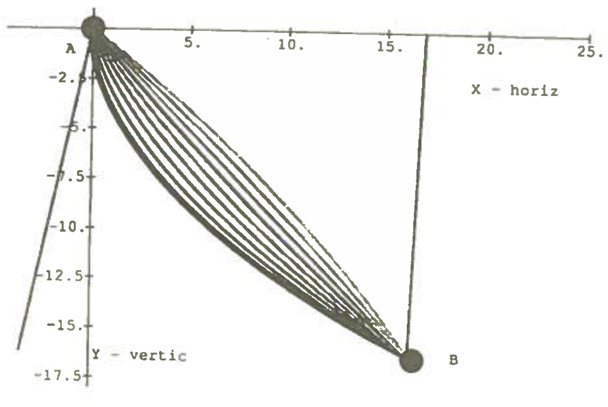

Figure 3

Figure 3: The family of trajectories $\{x_\varepsilon(\psi), y_\varepsilon(\psi)\}$ as seen from the observer, with the slope- and perspectival distortion.
From left to right: $\varepsilon = 0$ (brachistochrone: thick curve), $\varepsilon = 0.2$; $\varepsilon = 0.4$; $\varepsilon = 0.6$; $\varepsilon = 0.8$; $\varepsilon = 1$ (continuous segment AB); $\varepsilon = 1.2$; $\varepsilon = 1.4$; $\varepsilon = 1.6$.

The racers´ trajectories were video-recorded by an observer with a common device (25 pictures per second) of large focus length, located above the slope and far from the snow plane centre. The distortion of the original configuration as it appeared to the observer, due both to the slope angle and to the far distance of its position (perspective effect), is illustrated by Figure 3 on the same scale as in Figure 1 displaying the „true" map.
In order to scale spatially the video pictures, two columns of regularly spaced sticks (10m between two adjacent sticks) were placed on the slope at the left and right boundaries of the snow plane. These are the two grey boundary lines displayed by figure

3. The intermediate line is the perspectively transformed straight segment AB, while the lower thick curve is the perspectively-transformed optimal brachistochrone trajectory fitting the parameters **(1)** and illustrated by Figure 1. The thickness of this latter curve, equal to 0.3m, accounts for both the uncertainty of ± 0.5 degree about the angle $\theta = 26$ degrees of the slope (which leads to the ± 0.15m uncertainty in the localisation of the theoretical perspectively transformed cycloid curve AB) and the error of ± 0.5m in the dimensions of X and Y.

The racers were equipped with a special mark at their belts to indicate their centre-of-mass (CoM). We estimate the resulting error in the localisation of this CoM as ± 0.3m. It refers to the CoM position within the 10m interval between sticks, i.e. an error of likelyhood 3%, as well as to the subsequent laboratory analysis treatment process.

3. The Data Processing

The video monitor allowing a continuous range of picture display frequencies (from 25 pictures per second down to zero) was linked to an Olivetti M380.X1 personal computer whose software (Aissaoui, Blanchi and Roulet, 1987) allows with respect to some reference points an automatic output of the relative location of the skiers´ CoM appearing on the screen. The racers´ CoM were depicted on the monitor screen by use of two movable exploring cross-lines (horizontal and vertical). The error of such a measuring process is one pixel, which here means a horizontal and vertical error of ± 0.10m and ± 0.17m. These errors were added to the above-mentioned ± 0.3m error concerning the location of the CoM itself on the depiction of the skier´s body.

The data concerning the location of the skier´s CoM were periodically recorded on the hard-disc after each sequence of 10 pictures, i.e. in intervals of 0.4 seconds.

A second personal computer, Mc Intosh IIX, working with a Mathematics software (Wolfram et al., 1988), was superimposed upon the hard-copy output of the experimental CoM points. These theoretical test curves (with an automatic correction taking into account the perspectival and slope distortion effects - see Figure 3) approached them best.

In order to do so, we chose a family of test-curve functions depending on an adjustable parameter ε such that for the straight transverse segment AB $\varepsilon = 1$, while for the exact brachistochrone cycloid curve $\varepsilon = 0$. Its analytical expression reads:

$$x_\varepsilon(\psi) = R(\psi - \sin\psi) + \varepsilon \left[\frac{19.7}{37.1} R_{(1-\cos\psi)} - R_{(\psi - \sin\psi)}\right];\ y_\varepsilon(\psi) = -R(1-\cos\psi) \quad \textbf{(2)}$$

where R and ψ are the cycloid radius and the polar angle corresponding to the data **(1)**, i.e.:

$$R = 41.01\text{m} \pm 1.5\text{m and } 0 \leq \psi \leq 1.47\text{rd} \pm 0.03\text{rd} \quad \textbf{(3)}$$

These values are gained by numerically solving the system of equations **(2)** for the two particular points A (0,0) and B (19.7m ± 0.5m, 37.1m ± 0.5m).

The family of test-curves (2-3) is considered for $0 \leq \varepsilon \leq 1.6$. Figure 3 displays it as distorted by both:

- the slope-angle and
- the perspective-contraction effect.

The former effect is described by the slope-angle contraction factor: $m = \sin(26^0) = 0.438$ and only concerns the co-ordinate y. Clearly, an error of $\pm 0.5^0$ in the value of the slope angle θ results in an error of slightly less than ± 1% in the value of m, which scales the vertical size of the uncertainty concerning the localisation of the resulting curves as ± 0.15m. Therefore, *an error bar of 0.3m concerning the localisation of the theoretical curves on the slope and the perspectively transformed snow-planes must always be kept in mind when comparing them with the corresponding experimental data.*

The latter effect is measured by the „perspective-contraction factor“ $s(\psi)$. This factor is defined as the ratio - for each given value of the altitude co-ordinate y - of the distance which separates the two grey perspective boundary lines $a(\psi)$ and $b(\psi)$ shown by Figure 3 and directly recorded from the video data over the true downhill horizontal width of the plot AB, namely X = 19.7m. In the parametric description in terms of the polar angle ψ defined by Formulas **(3)** the two grey perspective boundary lines $a(\psi)$ and $b(\psi)$ read:

$$a(\psi) = \{-3.88(1-\cos\psi), -17.96(1-\cos\psi\}$$
$$b(\psi) = \{16.07 + 0.79\cos\psi, -17.96(1-\cos\psi)\}, \quad \textbf{(4)}$$

(two reference points were used from the video data in order to define each line).

As a consequence we numerically obtain the above-defined ratio s:

$$s(\psi) = 1.01 - 0.16\cos\psi \text{ (in meters)} \quad \textbf{(5)}$$

Then, in order to supply Figure 3, we multiply for each given value of the altitude co-ordinate y, $x_\varepsilon(\psi)$ (see equation **(2)**) by $s(\psi)$ and finally obtain from equations **(4)** the new „slope- and perspectively-transformed“ co-ordinates $x_\varepsilon^{new}(\psi)$ and $y_\varepsilon^{new}(\psi)$ for the family of curves **(2)** as:

$$x_\varepsilon^{new}(\psi) = s(\psi)\,x_\varepsilon(\psi) - 3.88(1-\cos\psi) = -3.88 + 22.05\varepsilon + 41.63\psi - 41.63\varepsilon\psi + 3.88\cos\psi - 25.47\varepsilon\cos\psi - 6.45\psi\cos\psi + 6.45\varepsilon\psi\cos\psi + 3.42\varepsilon\cos^2\psi - 41.63\sin\psi + 41.63\varepsilon\sin\psi + 6.45\cos\psi\sin\psi - 6.45\varepsilon\cos\psi\sin\psi,$$

$$y_\varepsilon^{new}(\psi) = mR(1-\cos\psi) = -17.96(1-\cos\psi) \quad \textbf{(6)}$$

For each member of the family of trajectories $\{x_\varepsilon(\psi), y_\varepsilon(\psi)\}$ described by equations **(2)**, the optimal theoretical racing-time value t_{AB} in the conservative case (i.e. no braking at all) was numerically calculated according to the following integral (Goldstein 1956; Reinisch 1990,1991):

$$t_{AB}(\varepsilon) = \int_A^B \sqrt{\frac{dx_\varepsilon^2 + dy_\varepsilon^2}{2Gy_\varepsilon}}$$

or by use of equations **(2)**:

$$t_{AB}(\varepsilon) = 4.53 \int_0^{1.47} \sqrt{\frac{\left[(1-\varepsilon)(1-\cos\psi) + 0.53\varepsilon \sin\psi\right]^2 + \sin^2\psi}{G(1-\cos\psi)}} d\psi \qquad (7)$$

where G is the effective gravity $G = 9.81$, $\sin(26.5^0)\ ms^{-2} = 4.38ms^{-2}$.

This function was then plotted versus ε and compared with the experimental data (see Figure 5 further below).

4. The Results

By superimposing upon the network of the theoretical curves displayed by Figure 3 the experimental results obtained through the above data processing procedure corresponding to the racers JR, MM and MRG we conclude Figures 4a,b,c (see page 79):

The points represent the positions of the racers´ CoMs at regular intervals (about 0.4 s). The extension of the points (0.6m) represents the uncertainty of the localisation of the racers´ CoMs. The figures 4a,b,c directly give the interval of ε-values corresponding to the approach of the trajectories of each racer by the family of curves displayed on Figure 3.

Let us consider, for instance, the case of JR: From Figure 4a we conclude that his „nearly-straight" trajectory was almost perfect, since the corresponding series of points all lie between the curves:

$$\left\{x^{new}_{[\varepsilon_{JR}=0.8]}(\psi), y^{new}_{[\varepsilon_{JR}=0.8]}(\psi)\right\} \text{ and } \left\{x^{new}_{[\varepsilon_{JR}=1.2]}(\psi), y^{new}_{[\varepsilon_{JR}=1.2]}(\psi)\right\} \qquad (8)$$

(remember that in all curves displayed by Figure 3 range - from bottom to top, from $\varepsilon = 0$ to $\varepsilon = 1.6$ by steps of $\Delta\varepsilon = 0.2$, various grey colours were used for clarity´s sake - the brachistochrone corresponds to $\varepsilon = 0$ while the straight segment AB is described by $\varepsilon = 1$).

We further read that Racer JR´s „downhill-curved" trajectory is considered to lie between $\varepsilon_{JR} = 0$ and $\varepsilon_{JR} = 0.4$.

Proceeding in the same way for the racers MM and MRG, we obtain from Figures 4b,c the intervals of ε-values corresponding to each of their runs:

Racer MM: $0.2 \leq \varepsilon_{MM} \leq 0.6$ and $0.8 \leq \varepsilon_{MM} \leq 1.2$;

Racer MRG: $0.6 \leq \varepsilon_{MRG} \leq 1.0$ and$1.0 \leq \varepsilon_{MRG} \leq 1.4$.

We now correlate these results with the corresponding racing-time values that have been measured during the experiment and we obtain the following concluding table:

Name	ε: curved traj.	ε: straight traj.	t_{AB}: curve	t_{AB}: straight
JR	$0.0 \leq \varepsilon_{JR} \leq 0.4$	$0.8 \leq \varepsilon_{JR} \leq 1.2$	4.58s ± 0.02s	4.81s ± 0.02s
MM	$0.2\ \varepsilon_{MM} \leq 0.6$	$0.8 \leq \varepsilon_{MM} \leq 1.2$	4.54s ± 0.02s	4.72s ± 0.02s
MRG	$0.6 \leq \varepsilon_{MRG} \leq 1.0$	$1.0 \leq \varepsilon_{MRG}\ 1.4$	4.64s ± 0.02s	4.75s ± 0.02s

Table 1

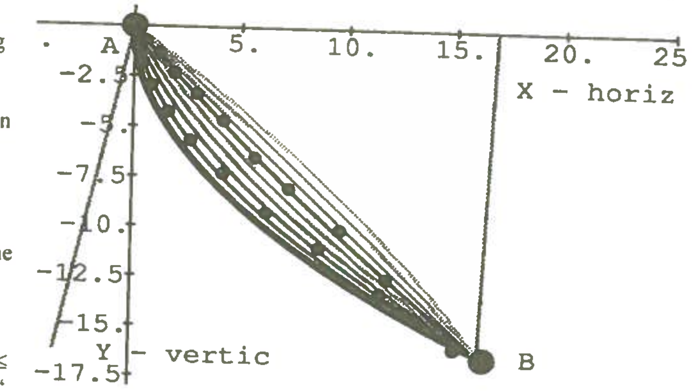

Figure 4a: The input of experimental results concerning Racer JR after his two runs, a „straight" one and a „downhill-curved" one, superimposed upon the family of theoretical curves displayed by Fig. 3. The extension of the points (0.6m), represents the uncertainty concerning the localisation of the centre-of-mass of the racer. We read on the plot that the „straight" trajectory may correspond to the interval $0.8\ \varepsilon \leq 1.2$, while the „downhill-curved" one can be regarded as lying between the ε-values 0.0 (brachistochrone) and 0.4.

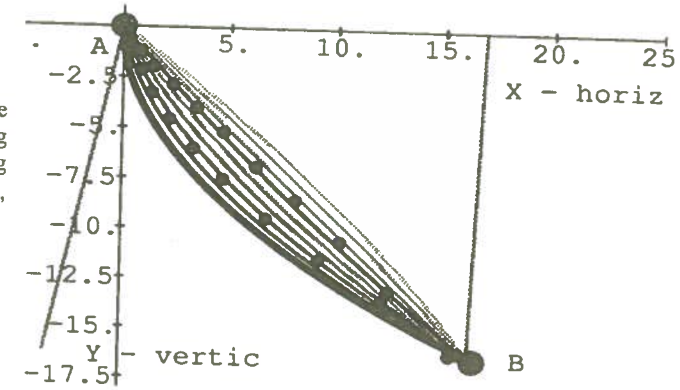

Figure 4b: Shows the same experiment as Figure 4a referring to Racer MM. The corresponding intervals of the ε-values are: (0.2, 0.6) and (0.8, 1.2).

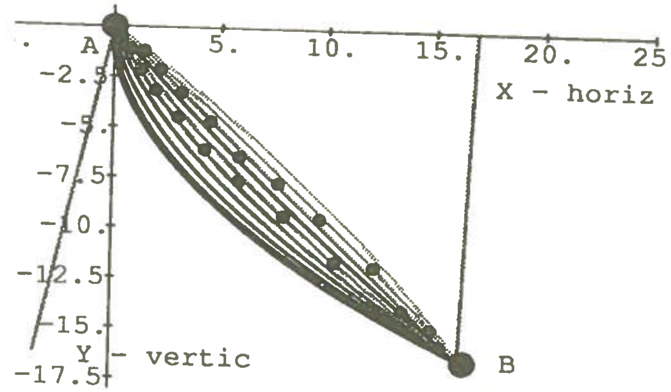

Figure 4c: The same figure as Figure 4a,b referring to Racer MRG. The corresponding intervals of ε-values are: (0.6, 1.0) and (1.0, 1.4)

Figure 4a, 4b, 4c

These values are now being plotted in the t_{AB}-versus-ε graph displayed by Figure 5. Three theoretical curves $t_{AB}(\varepsilon)$, corresponding (from bottom to top) to the values of the slope-angle $\theta = 26^0$, $\theta = 25.5^0$ (thick curve) and $\theta = 25^0$, have also been plotted according to Formula **(7)**.

Figure 5: The comparison between the experimental results of the Figures 4a, 4b, 4c as well as to Table 1 and the theoretical predictions corresponding to the slope angle θ equal 26 degrees (upper dashed curve), 26.5 degrees (thick middle curve) and 27 degrees (lower dashed curve). The experimental uncertainties concerning the measured racing-time values have been estimated at ± 0.02s. The light-grey, grey, and dark-grey rectangles concern the experimental data obtained from the runs of Racers JR, MM and MRG.

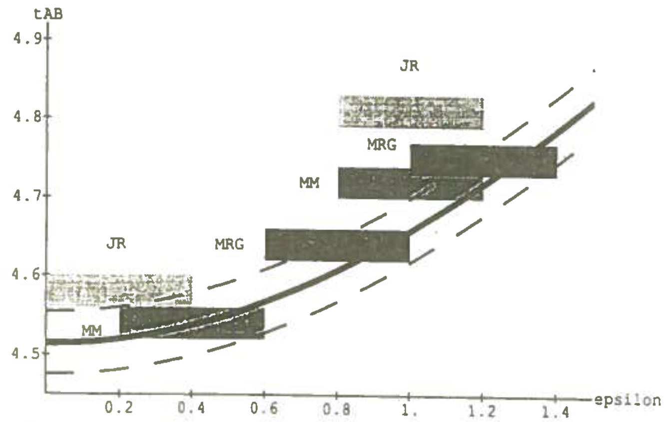

Figure 5

Obviously, the fit between the experimental values and the theory is quite close and acceptable when all uncertainity and/or error bars are taken into account. The racing-time along a slightly curved trajectory segment has been proved to be shorter than along the corresponding straight segment. The brachistochrone paradoxical effect has been verified in the context of top-level ski-racing.

5. Conclusion

Let us first point out a striking feature of the above results, which is quite obvious on Figures 4 a,b,c. Although it was left up to the racers to choose their trajectories AB and although they were told about the interest of a downhill bending of their trajectories just before the start of their second downhill-curved run, they did not deviate from the fall line as far as it would be required in order to describe the optimal brachistochrone curve. This means that *the brachistochrone effects are not reached intuitively, not even from high-level ski-racers.*

This point has to be emphasized with regard to the present discussions with national trainers and/or managers in alpine ski-racing, who may too often wish to get a personal feeling of such sophisticated topologic and dynamic effects by their intuition.

As it has been emphasized in the introduction of the present paper, the success of the theory concerning this particular case of *high-level ski-racers* (starting *at rest* and skiing along a *single curve* on a *uniform slope*) is a very strong argument in favour of the other predictions of the *same* theory, and in particular in favour of our prediction of a GS evolution towards the so-called Z-trajectory (Reinisch, 1990, 1991).

We have indeed been led to the idea that, sooner or later, such an evolution should lead to *highly discontinuous ultrashort turns about the gates*.

The problem of their feasibility implies (at least) two main investigations, which are of mutual interest:

i) a sport-biomechanical investigation of the possibility to resist the resulting huge and sudden increase of pressure distribution over the skier´s driving leg for highly trained young racers

ii) a ski-technical investigation of the criteria required in order to prevent the racer from losing more kinetic energy during these ultrashort turns than what he could gain during the preceeding flat-cycloid (quasi energy-conservative if the air resistance is neglected) transverse trajectory segment.

There are two aspects to this latter point:

iia) certain requirements of ski equipments (ski-boots, skis ...) to enable a highly discontinuous pressure distribution function with respect to time, including sharp „Dirac-function-like“ peaks;

iib) the pedagogical and training program to achieve the above declared results. This means the ability of and the possibility for ski-racers to „make ducks and drakes“ against two seperated parallel imaginary fall lines by the usual horizontal distance between two adjacent GS gates (about 20m). Clearly, such *rebounds* are, of course, quite realizable by high-level racers if their velocity is „low enough“. But what does „low enough“ quantitatively mean? Which is the threshold velocity allowing such abrupt ski-trajectory ricochets in the physical sense of *quasi-elastic rebounds*? Is it possible, by appropriate pedagogical and/or training exercises, to increase this threshold velocity?

Obviously, the present experimental study of the theoretically predicted brachistochrone paradox is only the very first step of a rich research program which is far from being merely academic: a single glance at Figure 5 shows that we are dealing with effects whose dimension is one tenth of a second. Who, trainers as well as managers being in charge of an international alpine ski-team, would deny the interest of such effects when „rowing hard in for a few hundredths of a second“ for the victory?

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