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A Physical Theory of Alpine Ski Racing

Abstract

The aim of this paper is the prediction of "time optimal trajectories" of the center of mass of an alpine ski racer who moves from an upper point *A* toward a lower point, *B* utilizing the fundamental laws of mechanics. The level of the racer is taken into account by introducing an implicit and an explicit damping term in the equations of motion (the former being described by its effects on the racer's velocity, rather than by an explicit analytical expression). We find that the optimal trajectory *AB* is always a part of a cycloide curve, which appears more or less flat, according to the initial velocity of the ski racer at point *A*. This curve is shown to be very close to a parabol which, in all realistic cases concerning giant slalom (GS), may actually be itself approximated by the portion of straight line *AB*. We conclude that the best trajectory tactics in GS, practically in all cases of slopes, consists in going straight to the gate pole and there turning "short-and-fast" when reaching the pole. This leads us to the conclusion that there should exist an "absolute optimal trajectory" – following a recent suggestion of the french ski trainer J. L. Monjo, we propose to call it the "Z-trajectory" – in which the above-mentioned short turns at the gates are replaced by basic unavoidable trajectory orientation discontinuities, which actually link the adjacent upper and lower linear trajectory segments.

Zusammenfassung

Ziel dieses Artikels ist es, die physikalischen Gesetze der Dynamik des Schwerpunktes zu erforschen, die einem Schifahrer erlauben, auf einem gleichförmigen Abhang und entlang einer „optimalen" Bahn die Distanz von A (oben) bis B (unten) in minimaler Zeit zu durchfahren. Dem Niveau des

Schifahrers wird durch einen Dämpfungskoeffizienten Rechnung getragen, implizit modulierbar (durch seine Wirkung auf die Geschwindigkeitskurve) oder explizit (durch eine analytische Formulierung des Koeffizienten in den Gleichungen der Bewegung). Die optimale Bahn A–B ist immer Teil einer Zykloide, mehr oder minder gestreckt, je nachdem ob die Ausgangsgeschwindigkeit bei A größer oder kleiner ist. Es wird gezeigt, daß sie mathematisch über einen Parabelteil approximiert werden kann, der in allen realistischen Slaloms und Riesenslaloms dem Geraden-Segment AB nahekommt. Daraus wird geschlossen, daß die beste „Fahrt-Strategie" für Slalom und Riesenslalom (und das auf allen Abhängen) in der Alltagssprache mit „Gerade–Ziehen – Kurz–Wenden" umschrieben werden kann. Es wird schließlich vorausgesagt, daß es eine „absolut optimale Bahn" gibt (wenn das Niveau des Schifahrers an Perfektion grenzt) – die nach dem französischen Skitrainer J. L. Monjo sogenannte „Z-Bahn" –, bei der die kurzen Kurven des „G–Z–K–W" durch noch kürzere unvermeidbare Richtungs-Diskontinuitäten ersetzt werden. Die Leistung des Slalom- bzw. Riesenslalomläufers wird durch Einhaltung der Z-Bahn grundlegend verbessert.

I – Introduction

The search of a powerfull tactics in alpine ski competition – and specially in giant slalom (GS), which is felt by many trainers and racers to be the fundamental event of all alpine disciplines <1> – is both an **unexpected** and a **stimulating** challenge for those physicists who are used to reduce a complex physical situation (which is *a priori* described by an large number of degrees of freedom) to a quite simple one (a mathematical low-dimensional system described by a few degrees of freedom).

"Unexpected" challenge in the sense that the traditional approach in alpine ski competition analysis is a highly particular heuristic one, based on a great amount of personal experience, in which the very numerous parameters related to *all* technical elements involved in ski racing are supposed to be *all* taken into account, according to what is believed to be their relative importance. Therefore, there is no serious attempt to reduce the number of degrees of freedom to a few pertinent ones. On the contrary, it seems that the present tendency lies in adding to a previous technical description and/or analysis of a given series of movements more and more sophisticated details <1, 2>. However, we feel that there is a lack of proper classification into relevant orders of magnitude.

As a matter of fact, the main trends among the alpine ski research concern either a detailed biodynamical analysis (ski and leg positions, lower and upper body positions and/or movements usually explained in terms of the balance of the acting forces <3>), or the account of the more and more advanced equipment technology (ski – and specially its chemical-physical processes allowing a better slipping –, shoes and their delicate balance between suppleness and rigidity a.s.o. <4>), or both <1–5>. This leads the "technicians" – i. e. the ski trainers having a solid theoretical background concerning these precise technical aspects – into an effort to add their main emphasis to:

i) the consistant body positions required for improving the balance and/or the efficiency in alpine ski racing, in accordance with the increasing physical capabilities of the athletes

ii) the carefull dissections of the turn in the considered ski event, say GS, with basically four phases – preparation, initiation, steering and completition –, taking each phase as a particular definite sequence of various technical movements which actually builds the bulk of the trainer's argument.

But what about the "best" trajectory linking, say, two adjacent turns, which is obviously the goal of the game and should result from a particular merger of all these phases into a coherent and harmonious flow of technical movements? Is it simply a **logical consequence** of an excellent ski technique – the so-called "good skiing" – allowing the perfect sequential performance of all the above phases, one to the next in a sort of markovian process? Or, on the contrary, is it the ground performance which should take the bearings of the evolution of all ski techniques? In other words, are there any indications which specify and uniquely determine such an optimal trajectory, **independently from the way each phase may be technically performed?**

In the former case, the "best" trajectory is not unique: it depends on the individual way of "good skiing" owned by each racer, although its foundations are more or less the same for all of them <6>. This is the widely shared point of view of the technicians up to now. On the other hand, the alternative case of a unique optimal trajectory – in given slope, snow and race course conditions – fixes the goal to be reached, and therefore might influence the technical aspects of each phase of the event. Its uniqueness would then fit with the popular feeling of "great skiing".

In alpine ski competition terminology, the experienced ski trainers know that there is a serious gap which separates "good skiing" – say, belonging to the group of the first hundred racers of the world – from "great skiing", basically the ten top ones. We believe that this gap might be related to the above discussion, which we now wish to translate in the following more physical question: **is there any elementary underlying mechanical principle – in the usual sense of dynamical systems – which should connect the various peculiar technical points of view of the trainers?** In other words, **does there exist a unified theory of GS racing tactics, at least in the limit of some idealistic cases?**

The aim of the present paper is to provide a positive answer to this question, and, surprising enough, we shall see that this answer has actually been given nearly exactly three centuries ago, by J. Bernouilli . . .

We said, the challenge is also **"stimulating"** for physicists, in the sense that on one hand the successes in alpine ski competitions are measured by extremely small time differences – a few hundreths of a second for races largely exceeding the minute. Therefore, any new discovery may give the corresponding ski team a substantial advance. On the other hand, a coherent scientific approach based on a few basic assumptions which are clearly emphasized and on a unifying description allowing the use of the mathematical-physical tools related to the Lagrange theory of dynamical systems is a pretty new attitude in this domain, and may open in the future a quite rich collaboration between ski technicians and theoretical physicists.

II – The concept of a perfect skier and the optimal brachistochrone trajectory

We adopt the following obvious definition: **a perfect skier is a conservative dynamical system.** He transforms his potential energy into kinetic energy without any losses.

The immediate question which naturally arises is the degree of idealization of such a concept with respect to ski racing reality. As a matter of fact, for racers belonging to the group of, say, the first hundred ski racers of the world – what we already called the "good-skiing" group –, their level, taking into account the high technology of their equipment, lies very close to this ideal state. **Good skiers, in the above sense, are actually quasi-conservative dynamical systems as long as they do not take a sharp turn, i. e. as long as they pilot their**

ski in an almost linear, transverse (with respect to the fall line) **or not, trajectory segment.** The detailed technical process allowing such a result consists, in the appropriate specialized terminology <7>, in maintaining a sufficient pressure on the downhill ski uphill edge in order to let this ski gently carve into the snow as opposed to "chattering" – this latter expression actually meaning damping –, and "cutting" the snow in order to "draw" the intended trajectory. The above description is only called up for those readers who have a good practice of alpine skiing, and it is out of the scope of the present paper to enter more deeply into such technical details. Let us simply emphasize that just such an apparently simple result is **not** easy to perform – it strongly depends on the edge angle between the steering ski and the snow, together with a delicate balance of all forces acting on the skier in such a way that the pressure on the ski is optimal with respect to its intended curve. When the snow is hard, and the slope is intermediate – 15 to 20 degrees –, the external conditions are best for such a performance.
We mention below that some recent experiments on snow did indeed corroborate the relevance of this concept of "perfectly skiing", when considering skiers with an even lower level than the above-defined "good skiing" one.
Consider a perfect ski racer who is ascribed to go within the shortest time interval from point *A* located uphill to point *P* located downhill – and not on the same fall line as *A*, in order to avoid a trivial case. Then it is actually well-known that the corresponding trajectory of the center of gravity of the perfect skier – which we shall always refer to as **the optimal trajectory of the equivalent particle** – is the solution of the "brachistochrone problem" solved in 1696 by J. BERNOUILLI. It consists of the portion of cycloid displayed by figure (1). Let us briefly remind the reader of its basic calculus <8>:

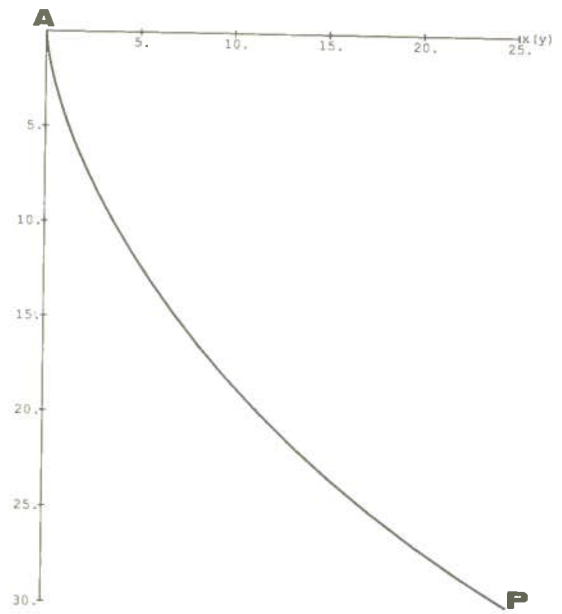

figure (1)

Let v be the particle velocity along the trajectory and v_A its initial velocity. The brachistochrone problem consists of finding the minimum value of the following integral:

$$t_{AP} = \int_A^P \frac{ds}{v} = \int_{y_A}^{y_P} \frac{\sqrt{1+x'^2}}{v(y)}\, dy \quad , \qquad (1)$$

where $s(t)$ is the particle curvilinear abcissa. The coordinates x and y are respectively the horizontal and the vertical abcissa of the skier center-of-gravity (the so-called "particle"), this latter being measured along – and in the direction of – the fall line of the slope. We note $x' = \frac{dx}{dy}$. For sake of simplicity, we always choose in the present work the origin of the axis at the "peak" of the cycloid (see figure [1]). Then $x_{A,P}$ and $y_{A,P}$ are the coordinates of the two bounds of the optimal trajectory.
The extremum of integral (1) yields by use of the Euler-Lagrange equation the following invariant since the horizontal coordinate x is cyclic:

$$\left[\frac{x'}{v(y)}\right] \sqrt{\frac{2g}{1+x'^2}} = constant = \Pi \quad . \qquad (2)$$

Here g is the effective value of the gravitation field: $G \sin \alpha$ (the angle α *is the slope angle and* G is the acceleration of the gravity).
Note that the constant Π entering formula (2) is understood with respect to the topological description $x(y)$ of the optimal trajectory.
From the conservation of energy written at the two points determined by $y = 0$ (Point A) and y:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_A + mg\,(y - y_A) \text{ with } \frac{1}{2}mv_A{}^2 = mgy_A \qquad (3)$$

we deduce the equation of the conservative cycloid brachistochrone trajectory:

$$\Pi = \frac{x'}{\sqrt{y\,(1+x'^2)}} \quad . \qquad (4)$$

The constant Π is a function of the radius R of this cycloide. We have indeed (see below equation [6]):

$$\Pi = \frac{1}{\sqrt{2R}} \quad . \qquad (5)$$

Recently, experiments on snow were performed by the American Ski Association, which verified that skiers, starting at rest and asked to follow such a cycloid trajectory, reached the arrival downhill point in a time interval less than when following the corresponding linear trajectory <9>. Similar expe-

riments need now to be performed with a larger spectrum of skier levels – in reference<9>, the skiers, although having an acceptable level, were not international racers –, and with a broad class of slope and snow conditions. However, the conclusions drawn from these existing experimental data are already of great importance, since they *a posteriori* confirm the relevance of the above-defined concept of a perfect skier. Consequently, they allow the use, at least in first approximation, of extremely simple physical tools such as the variational formulation of an integrable dynamical problem, leading to the skier center-of-gravity optimal trajectory.
The cycloid solution (4–5) is determined by the value of the radius R. This value depends on K and H which are respectively by definition the (absolute value of) the horizontal and vertical abcissa of the segment AP and on v_A which is the (absolute value of) the initial velocity (we remind that it is the velocity of the skier center of gravity – the "particle" – at A). As is well-known, the cycloid trajectory described by equation (4) may also be defined by the following parametric system of equations:

$$x = R\,(\theta - \sin\theta);\; y = R\,(1 - \cos\theta) \tag{6}$$

where θ is the cycloid polar angle. The radius R of this cycloide is then obtained by solving a transcendental system of three coupled equations in R, θ_A and θ_P, which depends on the race course parameters y_A, K and H:

$$y_A = R\,(1 - \cos\theta_A);\; K = R\,(\theta_P - \theta_A - \sin\theta_P + \sin\theta_A);\; H = R\,(\cos\theta_A - \cos\theta_P) \tag{7}$$

where y_A is defined by equation (3): $y_A = (v_A^2/2g)$. Inserting the mathematical formulation (6) into equation (4), we recover formula (5).
Equations (4–5) lead to:

$$x' = \frac{dx}{dy} = \pm\sqrt{\frac{z}{2 - z}} \quad , \tag{8}$$

where $z = y/R$. We obtain the solution $x(y)$ in an integral form:

$$x(y) = \pm R \int_{y_A/R}^{y/R} \sqrt{\frac{z}{2 - z}}\, dz \quad . \tag{9}$$

The $\pm$ sign which appears in formulas (8–9) obviously accounts for the left-right symmetry of the problem with respect to the fall line.
Note that the function $\sqrt{z/(2-z)}$ may be approximated, with an excellent accuracy, by the linear function 0,7842z+0,1848, for values of z lying between $z = 0,3$ and $z = 0,7$. Therefore, as long as this condition is valid – this restriction discards in particular the case of a skier starting at rest at A, since $z = 0$ at the cycloid peak –, **the optimal trajectory may be desribed by the following explicit parabolic function:**

$$x(y) = \frac{0.7842}{2R}\,(y^2 - \frac{v_A^4}{4g^2}) + 0.1848\,(y - \frac{v_A^2}{2g}) \quad . \tag{10}$$

The error consisting of approximating the original cycloide portion described by equation (9) by the parabola (10) is less than 4/1000 for values of R and v_A characteristic of practically all real situations in slalom and GS.
Note also that the slope of the snow field – i. e. parameter g – enters the expression of the optimal trajectory only through the lower bound y_A in the integral (9). Therefore, **for $v_A = 0$, i. e. for skiers starting at A at rest, the optimal trajectory is independant of the mountain slope.** Actually, this property approximately survives in most GS configurations, as shown by figure (2) where formula (10) has been used, since we took $v_A = 35\ km/h$.

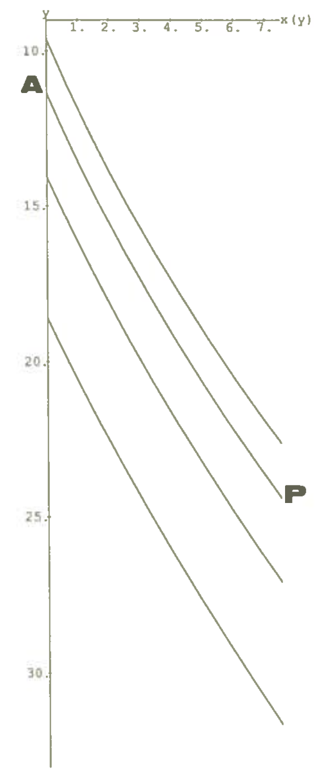

Figure (2)

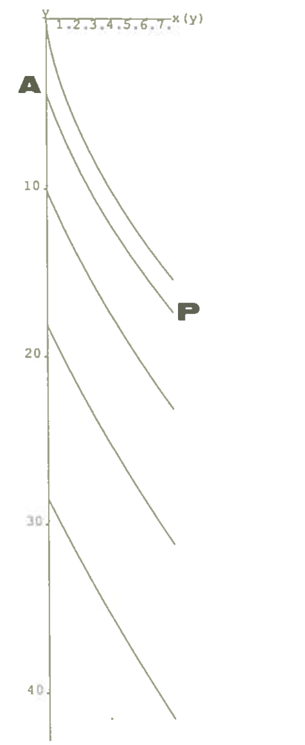

Figure (3)

Figure (3) displays the optimal trajectories for the same GS elementary configuration AP as in figure (2) – i. e. $K = 7,5\ m$; $H = 13\ m$ –, but for increasing initial velocities v_A. One notes that very soon – for $v_A \geq 30\ km/h$ – the optimal trajectory AP is rather linear. Therefore, **the same theory which ascribes to the optimal trajectory a substantial larger length than the linear seg-**

ment *AP*, when starting at rest (or at low initial velocities: see figure [3]), **tends to decrease this length discrepancy for higher velocities** – and actually, for values of v_A about *50 km/h,* the optimal trajectory lies very close to the linear segment *AP*. Since a real GS race consists in a sequence of several such *AP* parts performed at rather high velocities (about 60 km/h), one may now infer from these results that **"the best – i. e. optimal – way to get from one gate pole to the next one in GS is going rather linear.** This is actually the tactics up to date of most of the ten top racers in GS. But, as we pointed out <9, 10>, the grounds for ascribing to them such a tactics is far from the naive approach of trainers who, like the U.S. Ski Team Coach D. Bean, stated in reference <1> that *there is no argument against the fact the fastest way to get from Point A to Point B is a straight line . . .*

III – Taking short turns

Let us now consider the turns which link these quasi-linear (flat-cycloid) trajectory elements *AP* in order to build the whole GS race course. The dilemma we have to face is not simple: indeed, given a periodic sequence of such *AP* race course elements, the longer the quasi-linear optimal trajectory segments, the shorter the adjacent turns which follow, and as a consequence the greater the energy losses during these turns. Indeed **no skier may remain a conservative (or even a quasi-conservative) dynamical system when taking a short turn,** say typically of a radius less than 4 m. Therefore, which will be the average result of these contradictory effects on the energy balance?

Taking into account the high technology of the equipment together with the technique level of the top racers, it seems to us that the best compromise consists in getting from one GS gate pole to the next one as straight – or, better said now, as flat-cycloid – as possible and taking as short turns as possible, the lower limit for their radius of curvature lying about 2 m ref. <11> (see figure [4]).

This tactics consisting of "Going-Straight – Turning-Short" (GSTS) which we try to promote for the top class of GS racers <10, 11> spans indeed the future evolution of the top level in alpine ski racing. It means that the racers have enough biophysical and technical qualities to take short turns without loosing more energy than the energy amount gained during the quasi-conservative optimal trajectory state preceding the turns. Actually, the energy losses are even less than the gains at the first stages of the race, thus leading to the skier acceleration and to more and more linear transverse trajectory segments between the short turns. Then a stationary dynamical state is reached in which the energy losses (occuring mainly during the short turns) are balanced by the energy gains (occuring mainly during the flat-cycloid transverse trajectory segments). Such a stationary optimal trajectory is displayed by figure (4). The GSTS tactics will build the bulk of the next parts of the present work.

A. The conservative approximation

Although it is not quite realistic, as we pointed out in the above section, to consider a **global conservative situation** including the (short) turns, we first address this latter in as much as it spans the further discussion of the GSTS

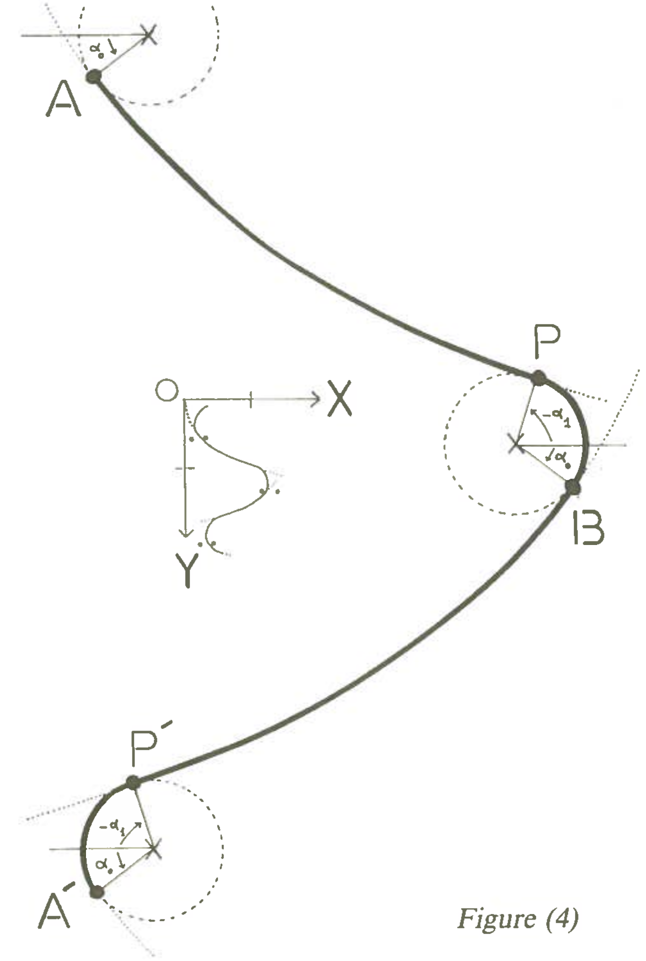

Figure (4)

tactics. Therefore, let us first stress the following conservative variational problem with constraints which is the *ad-hoc* generalization of the brachistochrone problem:

Consider a particle of mass unity falling, under the unique action of gravity, **in the shortest time interval** from point *A* to point *A'* lying on the same fall line as *A*, and assume the following additional conditions:

1. the particle trajectory must reach a third point, say, *B*, lying at an intermediate height with respect to *AA'* and not on the same fall line as *AA'* (see for instance figure [4]),

2. the trajectory and its first derivative dx/dy must both be continuous everywhere,
3. the values of the slopes $(dx/dy)_{A,\,A'}$ at points A and A' are given and assumed equal (see figure [4]),
4. the absolute value of the initial velocity at Point A, $(ds/dt)_A$ – where $s(t)$ is the particle curvilinear abcissa –, is given,
5. the local curvature radius $p(y)$ of the trajectory is always greater than a given minimum value $\rho_{min.}$

Each of the above constraints are the minimum conditions in order to avoid denaturing the problem with respect to the present state of alpine skiing competition. Indeed, constraint 1. allows the existence of a turn about point B in the trajectory AA'. Constraint 2. avoids discontinuity of the first and of the second kind (no angular points); the condition consisting of assuming that the first derivative (dx/dy) should be continuous will be dropped further below, when introducing the prospective concept of the so-called "Z-trajectory". Constraint 3. allows the definition of an **elementary GS course-mesh** entering the constitution of a **continuous periodic race course** (see figure [4] where the "spatial period" of the race course is measured by AA': the race course is assumed to be such that the trajectory starting at point A links with the trajectory ending at point A' a.s.o.). Constraint 4. is the obvious dynamical initial condition. Constraint 5. avoids singularities in the sense of the theory of distributions. This constraint will also be dropped when considering further below the prospective "Z-trajectory".

This variational problem has been solved by A. S. Fokas <2>. Its solution is such that the particle spends most of its time along the optimal cycloid trajectory parts AP and BP' (which are **not** equivalent, because of the conservative assumption, contrary to the case displayed by figure [4]), and tends to focus the above constraint 5. in a single turn of curvature radius equal to its minimal value ρ_{min}. This is quite realistic when keeping in mind the actual behaviour of top ski racers choosing now the GSTS tactics.

B. The non-conservative case

In order to introduce damping in the above sketch of the conservative optimal trajectory, we must first consider the perturbation to the cycloid parts AP and BP' due to existence of (small) energy losses.

a) the non-conservative brachistochrone problem

Equation (2) is interesting in the sense that it may be regarded to as a one-to-one mapping of any *a priori* given law $v(y)$ onto the set of optimal trajectories $x(y)_{opt}$. Stated in more intuitive terms, **there is always an optimal trajectory which corresponds to the level of a given racer.** Let us illustrate this statement by two simple examples:

i) the velocity $v(y)$ remains equal to the constant value v_A. This corresponds to a skier of rather poor level, choosing the tactics of uniform braking. Then, clearly, equation (2) yields $x'/(1+x'^2)^{1/2} = constant$, i. e. $x' = constant$, i. e. we recover the segment AP as the trivial optimal trajectory.

ii) the velocity v vanishes at a given value $y = y^*$. In this case the only non-singular solution of equation (2) implies $x'(y^*) = 0$. This explains the peak of the brachistochrone trajectory in the case of a particle starting at rest at point A.

From these two limiting cases, it becomes clear that a decrease of the skier velocity due to damping results into a downhill bending of the corresponding optimal trajectory which may balance – and even exceed – the cycloid uphill curvature. As a consequence, **damping usually leads to flattening the unperturbed optimal cycloid trajectory.**

b) the non-conservative heuristic model of taking turns and the corresponding optimal trajectory

Quantitatively speaking, this trajectory flattening may be illustrated in the following way. Assume a velocity distribution along y according to the following heuristic formula:

$$v(y) = \frac{1}{2}\left[\sqrt{2gy} + v_A + (\sqrt{2gy} - v_A)\,tanh\left[\frac{(v_A^2/2g) + H - 2r - y}{r}\right]\right], \quad (11)$$

which fits quite well the previous emphasis of top racers loosing energy only when taking short turns. Figure (5) displays such a function in the case of a GS defined by $K = 7{,}5\,m$ and $H = 13\,m$.

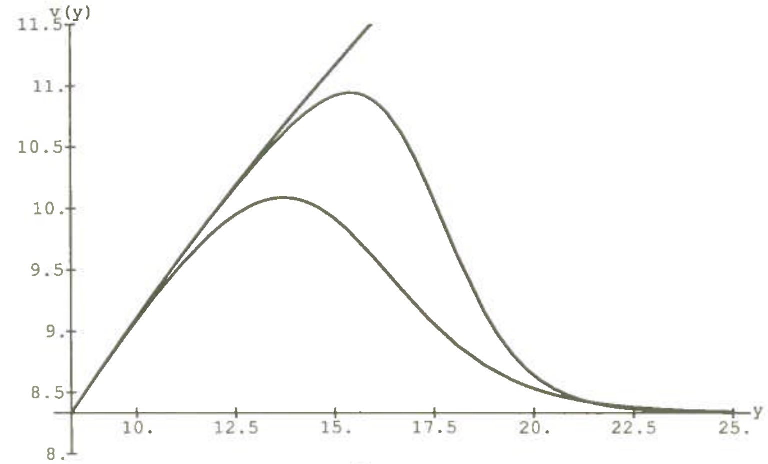

Figure (5)

The parameter r entering formula (11) is actually the "radius" of the considered turn. In this figure, we took $r = 2\,m$ and $r = 3\,m$. Then the general solution of equation (2) reads:

$$\frac{dx}{dy} = \pm\Pi\,\frac{v(y)}{\sqrt{2g - \Pi^2 v(y)^2}} \quad (12)$$

Therefore:

$$x(y) = \pm \Pi \int_{y_A}^{y} \frac{v(z)}{\sqrt{2g - \Pi^2 v(z)^2}} dz \quad . \tag{13}$$

The optimal trajectory (13) corresponding to the velocity distribution (11) is calculated by an iterative numerical process:
i) start the numerical integration of the trajectory (13) from the value of parameter Π, say Π_0 defined by equations (5) and (7) and corresponding to the unperturbed cycloid trajectory determined by the low *y-value* part of function (11) (see figure [5]).
ii) choose test values of Π about Π_0 such that the numerically integrated trajectory *x(y)* converges toward point *P* with the required accuracy. The result is displayed by figure (6), which corresponds to the velocity distributions shown by figure (5). The flattening of the unperturbed cycloid trajectory due to the local damping corresponding to the racer entering the short turn is quite clear.

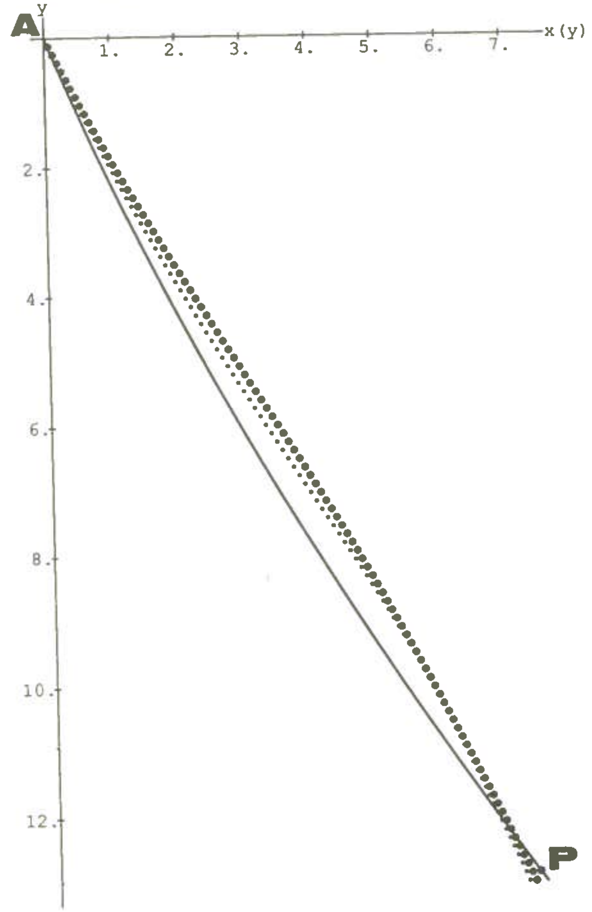

Figure (6)

IV – Dynamical grounds for GSTS tactics

Up to now, a summary of the main conclusions of the present study are the following:

a) in GS, at usual race course and velocity conditions, the quasi-conservative optimal trajectory from one gate pole to the next one is flat-cycloid-like, i. e. rather linear,
b) damping mainly occurs during the (short) turns and contributes to flattening the transverse cycloid trajectory elements,
c) the present state of the art in GS ski racing excludes any discontinuity of the trajectory derivative dx/dy.

Then a convenient dynamical model to check the relevance of the GSTS strategy consists of considering a family of trajectories basically built as a sequence of a circle segment (around the first pole), a linear segment and a second circle segment (around the next pole) similar to the first one (see figure [7]).

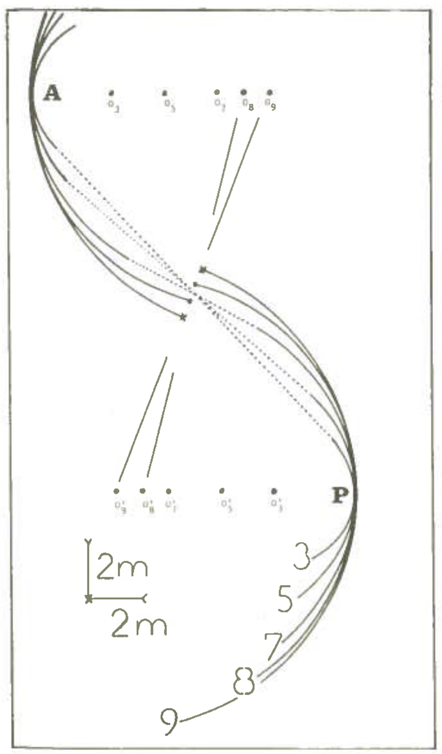

Figure (7)

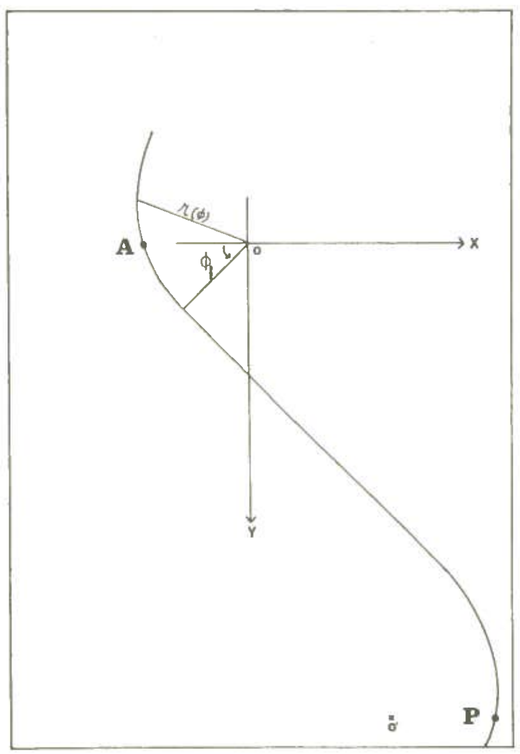

Figure (8)

The parameter labelling each member of this family is the circle radius r, according to:

$$D+2r = K \qquad (14)$$

where D and H are respectively the horizontal and vertical coordinates of point O'_r in the new frame displayed by figure (8).
Considering the most general case of a turn with a non-uniform curvature:

$$r(\phi) = r^*(1-\beta\phi) \quad where \quad 1-\beta(\frac{\pi}{2}) \geq 0 \quad , \qquad (15)$$

the effective lagrangian of the particle forced to move along a trajectory of the above family reads:

$$L = \frac{1}{2}m\left[(\frac{dr}{dt})^2 + r^2(\frac{d\phi}{dt})^2\right] - m g r \sin\phi + \lambda[r - r^*(1-\beta\phi)] \quad , \qquad (16)$$

where λ is the Lagrange multiplier related to the constraint (15). Writing the Euler-Lagrange equations corresponding to each degree of freedom r and $\varnothing$ and eleminating λ by use of equation (15) leads to the following equation of motion for the particle – the skier center-of-gravity – along the curve (15):

$$[r^2 + \beta^2 r^{*2}]\frac{d^2\phi}{dt^2} = g[r\cos\phi - r^*\beta\sin\phi] + r^*\beta r(\frac{d\phi}{dt})^2 \quad , \qquad (17)$$

where r is given by equation (15). Introducing in the standard way the **damping forces defined by the coefficient** α $(r^*, \varnothing)$ (which is *a priori* dependent on the position of the skier on the curve) yields:

$$\frac{d^2\phi}{dt^2} = \frac{g(r\cos\phi - r^*\beta\sin\phi) + r^*\beta r\phi_t^2}{r^2 + r^{*2}\beta^2} - \alpha(r^*,\phi)\phi_t \qquad (18)$$

The system of ordinary differential equations (15) and (18) is numerically integrated, using a 4th-order Runge-Kutta algorithm.

A. The uniform-curvature turn

Let us first consider the case $\beta = 0$. The integration of system (15), (18) requires:

a) the account of the proper initial and limit conditions
Considering figures (7) and (8), the uphill and downhill link points between the curved segment and the linear one have the respective coordinates:

$$x_{sup} = -r^*\cos\varnothing_f;\ y_{sup} = r^*\sin\varnothing_f$$
$$x_{inf} = D + r^*\cos\varnothing_f;\ y_{inf} = H - r^*\sin\varnothing_f \qquad (19)$$

Hence, the equation defining the linear segment tangenting both circle segments reads:

$$H\sin\varnothing_f - D\cos\varnothing_f = 2r^* \qquad (20)$$

The numerical solution of this transcendental equation with respect to $\varnothing_f$ gives the upper bound $\varnothing_f$ for the integration of the ordinary differential equation (18). Taking into account the peculiar symmetry of the family of trajectories displayed by figure (7), the system (15) and (18) is numerically integrated according to:

$$0 \leq \phi \leq +\phi_f : \left.\frac{d\phi}{dt}\right]_{t=0} = \frac{v_A}{r^*} \quad ,$$
$$-\phi_f \leq \phi \leq 0 : \left.\frac{d\phi}{dt}\right]_{t=t_2} = \frac{v_2}{r^*} \quad , \qquad (21)$$

where: $v_A = v\ (t = 0)$ is the curvilinear particle velocity at point A (i. e. the initial velocity), $v_1 = v\ (t = t_1)$ is the curvilinear particle velocity at the point defined by $\varnothing = \varnothing_f$, which is the link point between the uphill curved trajectory and the linear segment (see figure [8]), while $v_2 = v\ (t = t_2)$ is the instantaneous particle velocity value, reached at the link point between the linear segment and the downhill curve.
The linear section of the particle dynamics between both uphill and downhill curved trajectories may be analytically described, according to:

$$s(t) = \alpha^{-2} g\cos\phi_f(e^{-\alpha t} + \alpha t - 1) + \frac{v_1}{\alpha}(1 - e^{-\alpha t}) \quad , \qquad (22)$$

where we have considered the case of a uniform damping coefficient α $(r^*, \varnothing) = constant = \alpha$, which is realistic for the linear trajectory parts.
In order to obtain the value of the time-of-race t_3 corresponding to the whole trajectory AP, we must first (numerically) calculate v_2 (cf. the appropriate initial and limit condition [21]) and perform the following sequence of numerical procedures:
i) calculate t_1 as the numerical solution of the system (15) and (18) with the condition $\varnothing(t_1) = \varnothing_f$, already known from equation [20]),
ii) calculate t_2 from the time interval $t_2 - t_1$ (corresponding to the linear trajectory) obtained according to the following equation (cf. equation [22]):

$$s(t_2 - t_1) = \frac{D + 2r^*\cos\phi_f}{\sin\phi_f} \quad , \qquad (23)$$

where $\varnothing_f$ is given by equation (20),
iii) write (cf. equation [22]):

$$v_2 = \left[\frac{ds}{dt}\right]_{t=t_2} = \left[v_1 e^{-\alpha t} + \frac{g\cos\phi_f}{\alpha}(1 - e^{-\alpha t})\right]_{t=t_2} \quad , \qquad (24)$$

iv) then the final time t_3 and velocity v_3 corresponding to the particle reaching point P are obtained from the numerical solution of the system (15) and (18) with the conditions:

$$\phi(t_3) = 0 \quad ; \quad v_3 = r^* \frac{d\phi}{dt}\Big]_{t=t_3} \quad . \tag{25}$$

Let all damping coefficients vanish: then one should verify the conservation of energy

$$\frac{1}{2}(v_5^2 - v_0^2) = g\,H \tag{26}$$

which is therefore a convenient test in order to know the accuracy of the above sequence of numerical procedures i), ii), iii) and iv) which may be put in the form of a single global numerical program.

b) the heuristic definition of an acceptable damping coefficient $\alpha_c(r^*, \varnothing)$
Beside the trivial case of an uniform damping coefficient $\alpha\,(r^*, \varnothing) = \alpha_u$ which has already been considered above and is indeed realistic when the particle is describing the linear parts of its trajectory, we must envisage the additional case of an heuristic damping coefficient $\alpha_c(r^*, \varnothing)$ being all the more efficient as:
i) the curvature of the turn is higher (i. e. r^* smaller),
ii) the particle approachs the fall line while describing the turn. Let $v = v_{lim}(r^*, \varnothing)$ the limit velocity of the particle at the point of polar coordinates r^*, $\varnothing$ according to this damping. Since the particle driving force along the turn trajectory is $mgcos\varnothing$, we obtain:

$$\alpha_c(r^*, \phi) = \frac{g \cos\phi}{v_{lim}} \quad . \tag{27}$$

We now define $v_{lim}(r^*, \varnothing)$ by the following condition: the particle acceleration v^2_{lim}/r^*, scaled to $G = 9{,}81\ m/s^2$, equals a given dimensionless value μ (for instance, the choice $\mu = 1$ leads to $v_{lim} = 22{,}5\ km/h$ for $r^* = 4\,m$ which is reasonable, although slightly too pessimistic). Actually, $0 \leq \mu \leq 3$. Therefore, we obtain:

$$\alpha_c(r^*, \phi) = \frac{g \cos\phi}{\sqrt{\mu\, r^* G}} \quad . \tag{28}$$

Figure (9) displays the results corresponding to the family of trajectories defined by figure (7).
The time-of-race t_{AP} corresponding to the elementary course mesh from gate pole A go gate pole P is given in seconds. The lower curve corresponds to the conservative case $\alpha = 0$ everywhere. The medium curve corresponds to a small **uniform** damping coefficient $\alpha_u = 3\ 10^{-2} g\ (s/m)$ yielding **everywhere** the high limit velocity $v_{lim} = 33{,}3\ m/s = 120\ km/h$. The upper curve keeps this small uniform damping effect along the linear trajectory parts, but assumes the non-uniform damping effect defined by equation (28) in the turns. We chose the somewhat pessimistic value $\mu = 1$, in order to get an upper bound for the realistic damping effect occurring all along the trajectory AP. Thus the real damping should clearly concern the hachured area of figure (9) which is respectively bound from above and below by the choise of α_c and α_u.
Obviously, when looking at these results, the *Go-Straight – Turn-Short* tactics seems in any case efficient for top ski racers.

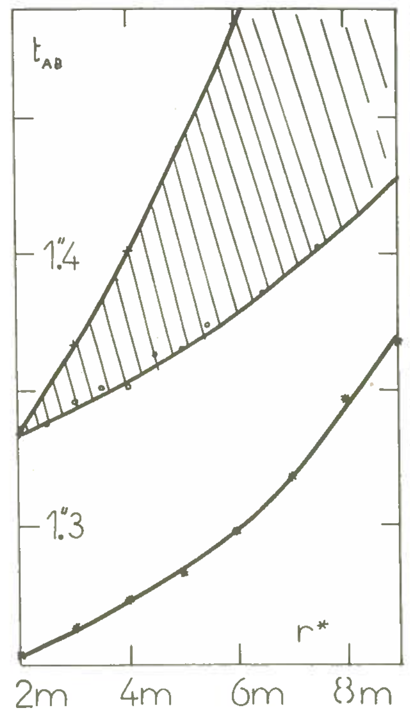

Figure (9)

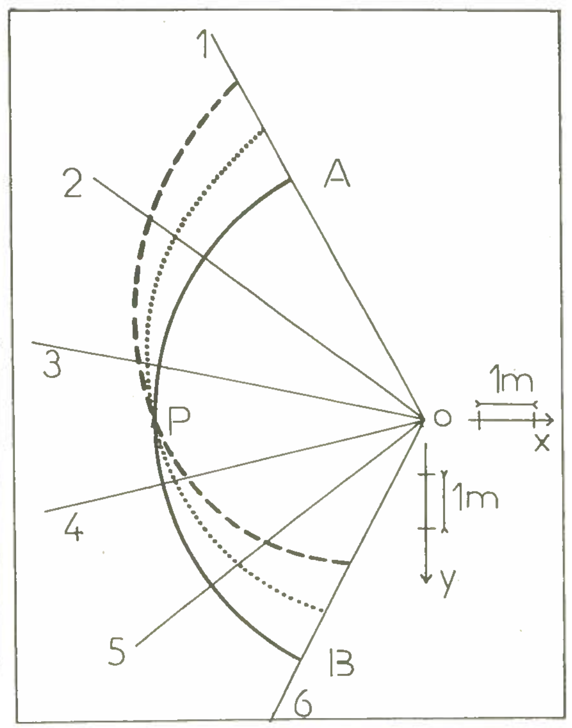

Figure (10)

B. The non-uniform-curvature turn

Ski trainers often wondered whether it is interesting for racers to increase the curvature of their turns about the beginning of the turns and then release it – in order to perform a sort of comma around the gate pole –, or to do the opposite (increase the turn curvature about the turn end, in a sort of inverted comma). This latter case, which corresponds to positive β values in equation (15), has been considered in figure (10).
Here A and B define the two **directions** which bound an angle of 120 degrees around the turn pole defined by point P, and β is measured by the value of $r(A) - r^*$, where r^* is the average value of the turn radius r (cf. equation [15]). From:

$$v = \sqrt{(\frac{dr}{dt})^2 + r^2(\frac{d\phi}{dt})^2} \quad , \tag{29}$$

we obtain

$$\frac{d\phi}{dt} = \frac{v}{r^* \sqrt{\beta^2 + (1 - \beta\,\phi)^2}} \quad . \tag{30}$$

Numerically integrating the ordinary differential equation (18) between $\varnothing = -\pi/3$ and $\varnothing = +\pi/3$ with the following initial condition:

$$\left.\frac{d\phi}{dt}\right]_{t=0;\phi=-\pi/3} = \frac{v_A}{r^* \sqrt{\beta^2 + [1 + (\pi\beta/3)]^2}} \quad , \qquad (31)$$

leads to the results displayed by figure (11), where $0 \leq \beta \leq 3/2\pi$ $(r^* = 5\,m)$ and $0 \leq \beta \leq 7{,}5/8\pi$ $(r^* = 8\,m)$. The lower curves correspond to the conservative case $\alpha = 0$, while the upper curves display the case $\alpha = \alpha_c$ with $\mu = 1$ (cf. equation [28]).

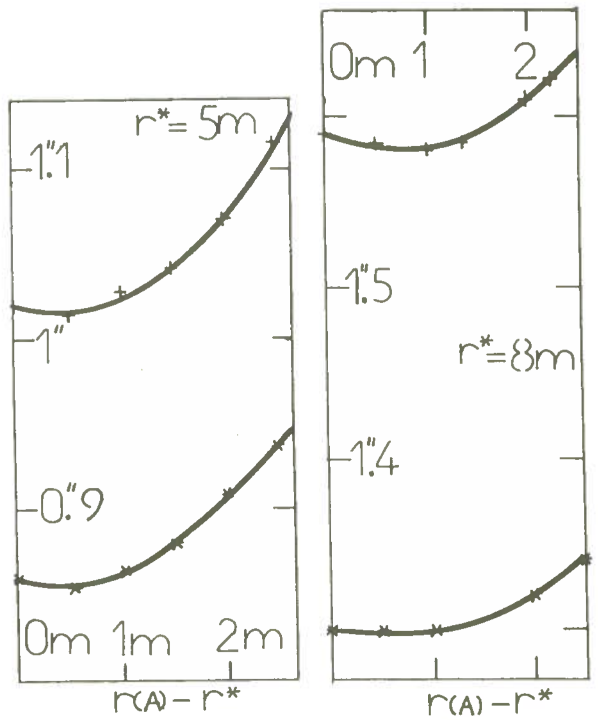

Figure (11)

The comma-like turns (negative *β-values) were also considered and led to similar results. Hence, we conclude that* **uniform-curvature turns seem best appropriate for ski racing, within the frame of the present model.**

V – Conclusion: towards the "Z"-trajectory?

Although the present state of the art in alpine ski race "coaching" lies pretty far from the above physical considerations, it may be of interest to proceed one step further towards theoretical predictions and display **the optimal trajectory for given slope, snow and race course conditions in the ideal case of a perfect skier** (i. e. of a conservative dynamical system).

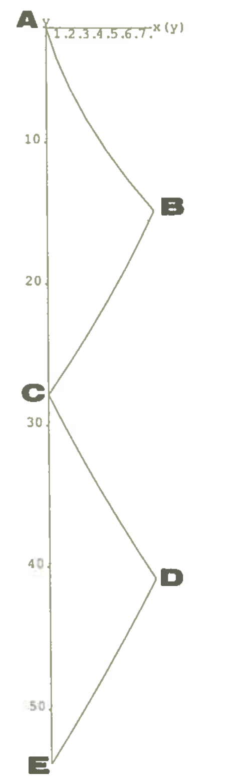

Figure (12)

Figure (12) displays such a trajectory in the case of a race course defined by $K = 7{,}5\ m$; $H = 13\ m$ on a slope of 25 degrees and with a (small) initial velocity equal to $v_A = 15\ km/h$. Due to energy conservation, the successive velocities at the poles *B, C, D, E* respectively are: 40,29 km/h, 54,97 km/h, 66,48 km/h, and finally 76,28 km/h and, accordingly, the optimal cycloid-like trajectory segments are more and more linear. This optimal piecewise differentiable trajectory is simply the solution of the variational problem stated in II-A, when constraint $n^o\ 2$ concerning the continuity of the derivative dx/dy is discarded: then the optimal solution, defined in the functional space inclu-

ding distribution functions is clearly a continuous sequence of cycloid elements such as displayed by figure (12).
These velocity values are obviously excessive. However, the "experiment" is possible: set a **very regular (i. e. periodic)** GS race-course on a uniform slope and take instantaneous time-of-race values, velocity values and pictures at different points of the trajectories which are effectively followed by racers. Various ski technique levels – measured in the so-called *Fédération Internationale de Ski (FIS) points* – are of course important parameters of such an experiment. Concerning a given sequence of, say, five such turns as shown by figure (12), the following non-exhaustive list of questions should be addressed: i) if the racers are asked to start at point A at rest, how far does the first trajectory segment AB lie from the optimal brachistochrone cycloid curve which, given points A and B, may easily been numerically superimposed to the trajectory pictures taken on the slope? ii) does this first trajectory segment converge towards the cycloid as a function of the ski racer level, and how? iii) do the subsequent trajectory segments BC, CD, DE... indeed become flatter and flatter, according to the flattening of the optimal cycloid trajectory as a function of the increasing particle velocity, and how? iv) how do the instantaneous time-of-race and velocity values measured at points B, C, D, E... differ from their idealistic theoretical values corresponding to the conservative case? v) by introducing damping according to the test formula (28) in the equation of motion (18), is it possible to explain these experimental time-of-race values by use of a rather systematic value of parameter μ, and what is the acceptable range for μ-values?
There is a sixth question which we would like to emphasize as one of the thesis of the present work concerning the prospective evolution of alpine ski tactics: **isn't it possible for top ski racers to perform at a given pole point P, first in very peculiar "experimental" conditions (i. e. in particular race course, velocity, snow and slope training conditions), and then in some *ad-hoc* racing conditions a discontinuous trajectory-derivative jump from the value $+|dx(y)/dy|_p$ to the value $-|dx(y)/dy|_p$ which respectively correspond to the optimal trajectory derivative value immediately before and immediately after the pole P?**
The author does indeed think that the present tendency of top ski racers to shorten the turns and straighten the trajectory segments between them, according to the GSTS tactics, is the first step of an evolution which actually leads towards the above-described piecewise continuous "Z-trajectory" as illustrated by figure (12). Indeed, such a trajectory obviously reduces to its minimum duration – in a sort of Dirac peak of energy loss – the (unavoidable) damping due to a sudden change of trajectory orientation and makes maximum the extension of the (quasi-conservative) cycloid-like transversal segments.
The first test which such a future tactics will have to encounter is concerned by the average energy balance: can a ski racer perform such a very tonic and sharp trajectory jump about the gate poles of a given GS without loosing more energy than when describing a continuous – although short – turn around these poles? Or, better said: is the average energy of the skier-center-of-mass higher after such a given sequence consisting of a long transversal quasi-cycloid trajectory segment and a subsequent trajectory orientation jump than after a shorter cycloid segment and a subsequent continuous (although quite short) turn around the pole?
The author believes that providing a definite answer to these questions should bulk large in the technician's eyes when trying to define a long-range prospective theoretical and experimental ski training program for World-Cup racers.

Figure captions

Figure 1:

The cycloid optimal trajectory of the skier center-of-mass starting at rest at point A and reaching point P within the shortest time-of-race. The coordinates of the segment defined by the couple of points A, P are: $K = 24{,}5\ m$; $H = 30\ m$. The scales of the two axis are identical.

Figure 2:

The optimal trajectories AP respectively corresponding to the skier center-of-mass starting at point A with the velocity $v_A = 35\ km/h$ and to the following slopes (respectively from bottom to top): 15 degrees (weak slope), 20 degrees, 25 degrees, 30 degrees (hard slope). The values of x and y are given in meters and the scales of the two axis are identical. Each trajectory correspond to the same segment defined by the couple of points A, P according to: $K = 7{,}5\ m$, $H = 13\ m$. The position of point A on the vertical y-axis corresponds to $y_A = v_A^2/2g$ (cf. equation [3]).

Figure 3:

The optimal trajectories AP obtained under the same conditions as for figure 2, but for decreasing initial velocities v_A respectively equal to (from bottom to top): 50 km/h, 40 km/h, 30 km/h, 20 km/h, and 0 km/h (i. e. starting at rest). The slope is 20 degrees for all cases.

Figure 4:

Race-course mesh built by a sequence of two flat-cycloid transverse trajectory elements separated by a short turn whose curvature radius equals 2 m. A stationary dynamical regime is assumed, in which the energy gain obtained after each flat-cycloid transverse trajectory element AP or BP' is balanced by the energy loss due to the short turn PB or $P'A'$. The trajectory orientations dx/dy at points A and A' are equal in order to allow the periodicity of the race course. The angles α_0 and α_1 are simply related to the cycloid polar angles θ_A and θ_P: $\alpha_0 = \theta_A/2 = \pi/5$ and $\alpha_1 = -\theta_P/2 = -2\pi/5$. The cycloid radius is

$R = 15\,m$, corresponding to the gate poles A and B located according to: $K = 14\,m$ and $H = 12\,m$. The initial velocity is $v_A = 22\ km/h$ and the slope is 20 degrees.

Figure 5:

Three different velocity distributions along the fall line, according to the value of the short turn radius r (cf. formula [11]). From bottom to top: $r = 3\,m$, $r = 2\,m$, and $r = 0\,m$ for a GS race-course mesh corresponding to $K = 7{,}5\,m$ and $H = 13\,m$ on a slope of 25 degrees ($g = 4{,}15\ m/s^2$). The initial velocity v_A is 30 km/h. The upper curve corresponds to the low-y part of the conservative velocity distribution $v = \sqrt{2gy}$ and allows comparison with the heuristic function described by formula (11). Units are in m (for y) and in m/s (for v).

Figure 6:

Optimal trajectories corresponding to figure 5. The scales of both axis (in meters) are identical. Continuous line: the reference cycloid ($\theta_A = 0{,}77\ rd$; $\theta_P = 1{,}28\ rd$; $R = 29{,}76\ m$) corresponding to the conservative case (upper curve of figure 5). Thin dashed line: $r = 2\,m$ (medium curve of figure 5). Thick dashed line: $r = 3\,m$ (lower curve of figure 5). The small uncertainty related to the use of an iterative numerical integration process concerning formula (13) is obvious when considering the location of point P which, in the present frame, reads: $K = 7{,}5\,m$ and $H = 13\,m$. We obtained by successive iterations the values $1/2\Pi^2 = 24{,}25\ m$ for $r = 2m$ and $1/2\Pi^2 = 21{,}49\ m$ for $r = 3\ m$ (see formula (5)), where as the "unperturbed" value of the cycloid radius (corresponding to the continuous line trajectory) is $R = 29{,}76\,m$.

Figure 7:

The family of elementary race-course trajectory meshes corresponding to different GSTS tactics choices (according to the value of the short turn radius r equal to 9 m, 8 m, 7 m, 5 m and 3 m) around the two gate poles A and P. We have $K = 12\,m$; $H = 15\,m$. The initial velocity v_A is assumed equal to 50 km/h.

Figure 8:

The geometrical frame measuring the position of the skier center-of-mass in the GSTS tactics.

Figure 9:

The time-of-race t_{AP} corresponding to the family of trajectories displayed by figure 7. From bottom to top: the conservative case (no damping at all); the uniform damping case according to $\alpha = 3\ 10^{-2}$g (s/m); the non-uniform damping case described by formula (28) with $\mu = 1$. A reasonable estimate of the dynamical effects concerning **real** damping forces is believed to lie in the hachured area.

Figure 10:

The non-uniform-curvature turn. Large-dashed trajectory line: $r(A) - r^* = 2\,m$; thin-dashed trajectory line: $r(A) - r^* = 1\ m$; continuous trajectory line: $r(A) = r^* = 5\ m$ (cf. formula [15]). Each segment labelled by a number defines an additive angle of 24 degrees.

Figure 11:

The time-of-race t_{AP} corresponding to the family of non-uniform-curvature turns displayed by figure 10 with $r^* = 5\,m$ (left-hand-side diagrams) and with $r^* = 8\,m$ (right-hand-side diagrams). The initial velocity v_A is 40 km/h. Lower curves: no damping (conservative case); upper curves: the non-uniform damping case described by formula (28) with $\mu = 1$.

Figure 12:

The optimal "Z" trajectory in the idealistic conservative case, displayed in a frame of same vertical and horizontal scale (units are given in meters). The elementary race-course trajectory meshes AB, BC, CD and DE are all identical in order to build **a periodical** race course. They are defined by $K = 7{,}5\,m$ and $H = 13\,m$. The slope is equal to 25 degrees. The instantaneous velocities are: $v_A = 15\,km/h$; $v_B = 40{,}29\,km/h$; $v_C = 54{,}97\,km/h$; $v_D = 66{,}48\,km/h$, and $v_E = 76{,}28\,km/h$. Note the steepening of the optimal transverse cycloid trajectory elements as the velocity increases.

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KLAUS CACHAY

Sportpädagogik und Gesellschaftstheorie

Zusammenfassung

Der Beitrag geht von der Annahme aus, daß die Aufgabe der Pädagogik mit der Beschreibung der „Erziehungstatsache" beginnt. Zur Konstruktion dieser „Erziehungstatsache" wurden in der Sportpädagogik bislang vornehmlich Subjekttheorien verwendet, Gesellschaftstheorien dagegen weitgehend ausgeklammert. Dies führte dazu, daß sich die Sportpädagogik gesellschaftlichen Problemen aus der Perspektive von Lernprozessen kaum zugewendet hat. Am Beispiel des Sport-Umwelt-Konflikts wird versucht, die Relevanz gesellschaftstheoretischer Überlegungen für die Sportpädagogik aufzuzeigen.

Abstract

The article proceeds on the assumption that the purpose of pedagogy begins with the description of "educational fact". The definition of "educational fact" has, until now, essentially consisted of theories of subject, whereas social theory has been predominately avoided. As a result, the pedagogy of physical education has hardly dealt with problems of society from the perspective of learning processes. Based on the example of the conflict of sport and environment, the relevancy of social theory for physical education is herein demonstrated.